

On two simple criteria for recognizing complete intersections in codimension 2

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Abstract

Developing a previous idea of Faltings, we characterize the complete intersections of codimension 2 in \mathbb{P}^n , $n \geq 3$, over an algebraically closed field of any characteristic, among l.c.i. X , as those that are subcanonical and scheme-theoretically defined by $p \leq n - 1$ equations. Moreover, we give some other results assuming that the normal bundle of X extends to a numerically split bundle on \mathbb{P}^n , $p \leq n$ and the characteristic of the base field is zero. Finally, using our characterization, we give a (partial) answer to a question posed recently by Franco, Kleiman and Lascu ([4]) on self-linking and complete intersections in positive characteristic.

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1 Introduction

The problem of detecting (global) complete intersections is a key question in projective algebraic geometry and commutative algebra. Up to now, this problem is far from being solved and a complete answer is known only in trivial cases, such that of hypersurfaces in projective spaces or in Grassmannians. Moreover, the relevant conjecture of Hartshorne according to which projective subvarieties of small codimension, compared to their dimension, have to be complete intersections, is still not proved. However, in the last 25 years, there have been some partial results in this direction, particularly in the case of codimension 2. Essentially, the results obtained in the case X is a smooth codimension 2 subvariety of \mathbb{P}^n , $n \geq 6$ can be grouped into two kinds of criteria.

The first one says that if X is contained in a hypersurface V , such that $\deg(V) \leq n - 2$, then X is a complete intersection (see [8] or the recent improvement in [2], where it is shown that the bound on the degree of V can be increased to $n - 1$, in the case of codimension 2 subvarieties of \mathbb{P}^6); using this kind of criterion one can give also a bound on the degree of X , so that to assure that X is a complete intersection.

The second kind of criterion is based on giving a bound on the number p of generators, not for the homogeneous ideal $I(X)$, but for an ideal $I_{sch}(X)$ which coincide with $I(X)$ only in high degree, that is $[I_{sch}(X)]_d = [I(X)]_d$, for $d \gg 0$. We call $I_{sch}(X)$ the schematic ideal of X , in that its generators define X scheme-theoretically. Following this approach, Faltings proved in [3] that if $p \leq n - 2$ and $n \geq 8$ and X is a (possibly singular) subcanonical local complete intersection, then it is a complete intersection (in any characteristic). Some years later, this result was improved in [7], proving in characteristic zero that if $p \leq n - 1$, $n \geq 8$, then X is a complete intersection (but assuming X smooth).

The aim of the present work is twofold: on one hand we would like to give a different (in that we use Serre's correspondence) and simpler proof of the result announced in [7], hoping to give a crystal clear version of some obscure (to our opinion) arguments. Moreover, assuming only that X is a (possibly singular) subcanonical l.c.i., we prove in any characteristic that if $n \geq 3$ and $p \leq n - 1$, then X is a complete intersection and we give some more results working in characteristic zero, assuming that the normal bundle of X extends to a numerically split bundle E on \mathbb{P}^n (i.e. the Chern classes of E are those of a split bundle), $n \geq 3$, $p \leq n$. On the other hand, as an application of our

result we answer to a question posed recently by Franco, Kleiman and Lascu in [4], (neglecting the case of space curves). Unfortunately, our result shows that the characterization given by Faltings *is not peculiar* of two codimension embeddings in high dimensional projective spaces.

Our proof is based in constructing and exploiting an exact sequence of locally free sheaves, (sequence (3)), which relates the rank 2 vector bundle E appearing in Serre's correspondence with the generators of the scheme-theoretic ideal of X .

2 Main result: a scheme-theoretic criterion

From now on, X will denote a codimension 2 *subcanonical* l.c.i. (possibly singular) closed subscheme of a projective space \mathbb{P}^n over an algebraically closed field k of any characteristic $p \geq 0$, where, as usual $\mathbb{P}^n = \text{Proj}(k[x_0, \dots, x_n])$. Then we prove the following result:

Theorem: *A) If $X \subset \mathbb{P}^n$, ($n \geq 3$) is a scheme-theoretic intersection of $p \leq n - 1$ hypersurfaces, then X is a complete intersection.*

B) Assume moreover that $\text{char}(k)=0$, $n \geq 3$, and X scheme-theoretically defined by $p \leq n$ hypersurfaces V_1, \dots, V_n of degrees d_1, \dots, d_p , respectively. If the normal bundle of X can be extended to a rank 2 vector bundle E on \mathbb{P}^n which is numerically split (i.e. $c_1(E) = a + b$ and $c_2(E) = ab$, $a, b \in \mathbb{Z}$) and a or b is in (d_1, \dots, d_p) , then X is a complete intersection.

Proof of part A: Since X is assumed to be subcanonical (i.e. its dualizing sheaf ω_X , which is locally free, is of the form $\mathcal{O}_X(e)$), by Serre's correspondence there exists an algebraic vector bundle E of rank 2 over \mathbb{P}^n and a section $s \in H^0(\mathbb{P}, E)$ such that X is identified with the scheme of zeroes of s , $Z(s)$. The Koszul complex for this section gives a projective resolution of the ideal sheaf of $Z(s)$, hence of the ideal sheaf of X :

$$0 \longrightarrow \bigwedge^2 E^* \longrightarrow E^* \longrightarrow \mathcal{I}_X \longrightarrow 0. \quad (1)$$

Since \mathcal{I}_X is not itself projective, by (1) it turns out that the projective dimension of \mathcal{I}_X is 1. On the other hand, if X is schematically cut out by $p \leq n - 1$ hypersurfaces of degrees d_1, \dots, d_p , we have an exact sequence:

$$0 \longrightarrow \text{Ker}(f) \longrightarrow \bigoplus \mathcal{O}(-d_i) \xrightarrow{f} \mathcal{I}_X \longrightarrow 0. \quad (2)$$

Since $\text{pd}(\mathcal{I}_X)=1$, then the first syzygy $\text{Ker}(f)$ is also projective (see for example [9]), hence it corresponds to a locally free sheaf. Certainly, we have a morphism $h \in \text{Hom}(\oplus \mathcal{O}(-d_i) \oplus \mathcal{O}(-c_1), \mathcal{I}_X)$ which is given first by projecting to $\oplus \mathcal{O}(-d_i)$ and then composing with f , (c_1 is the first Chern class of E). Moreover, since $\text{Ext}^1(\oplus \mathcal{O}(-d_i) \oplus \mathcal{O}(-c_1), \mathcal{O}(-c_1)) = 0$, then h comes from an element g in $\text{Hom}(\oplus \mathcal{O}(-d_i) \oplus \mathcal{O}(-c_1), E^*)$; indeed, due to the fact that $\Lambda^2 E^* \equiv \mathcal{O}(-c_1)$, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(-c_1) & \longrightarrow & \oplus \mathcal{O}(-d_i) \oplus \mathcal{O}(-c_1) & \longrightarrow & \oplus \mathcal{O}(-d_i) & \longrightarrow & 0 \\ & & \downarrow \equiv & & \downarrow g & & \downarrow f & & \\ 0 & \longrightarrow & \Lambda^2 E^* & \longrightarrow & E^* & \longrightarrow & \mathcal{I}_X & \longrightarrow & 0 \end{array}$$

Applying the snake lemma to the previous commutative diagram, we see that g is surjective and that $\text{Ker}(g) \equiv \text{Ker}(f)$, so that dualizing the sequence $0 \longrightarrow \text{Ker}(g) \longrightarrow \oplus \mathcal{O}(-d_i) \oplus \mathcal{O}(-c_1) \xrightarrow{g} E^* \longrightarrow 0$, we get a short exact sequence of *locally free* sheaves:

$$0 \longrightarrow E \xrightarrow{\alpha_1} \oplus \mathcal{O}(d_i) \oplus \mathcal{O}(c_1) \xrightarrow{\alpha_2} C \longrightarrow 0, \quad (3)$$

where C is just the cokernel sheaf. Since C is locally free, it can be identified with a vector bundle of rank equal to $p - 1$ ($p \leq n - 1$). Let in_j denote the canonical injection of $\mathcal{O}(d_j)$ into $\oplus \mathcal{O}(d_i) \oplus \mathcal{O}(c_1)$, and pr_j the corresponding projection from $\oplus \mathcal{O}(d_i) \oplus \mathcal{O}(c_1)$ to $\mathcal{O}(d_j)$. Considering the maps $f_j := \text{pr}_j \circ \alpha_1$ and $g_j := \alpha_2 \circ \text{in}_j$ we have a diagram like the following:

$$\begin{array}{ccccccc} 0 & \longrightarrow & E & \xrightarrow{\alpha_1} & \oplus \mathcal{O}(d_i) \oplus \mathcal{O}(c_1) & \xrightarrow{\alpha_2} & C \longrightarrow 0 \\ & & \searrow^{f_j} & & \text{pr}_j \downarrow \uparrow \text{in}_j & \nearrow^{g_j} & \\ & & & & \mathcal{O}(d_j) & & \end{array}$$

Now consider the morphisms $E \xrightarrow{f_1} \mathcal{O}(d_1)$ and $\mathcal{O}(d_1) \xrightarrow{g_1} C$ and denote by $Z(f_1)$ and $Z(g_1)$, their respective degeneracy loci. Since in general we have that $\text{codim}(Z(f_j)) \leq 2$ and $\text{codim}(Z(g_j)) \leq p - 1$, it turns out that if $p \leq n - 1$, then $Z(f_1) \cap Z(g_1) \neq \emptyset$. On the other hand, by exactness of (3), it is clear that $Z(f_1) \cap Z(g_1) = \emptyset$. Indeed, if it exists $x \in Z(f_1) \cap Z(g_1)$, then $\text{Ker}(f_1)_x = E_x$, but since the morphism α_1 can not degenerate at any point (due to the fact that the cokernel C is locally free), we have that $\text{Im}(f_1)_x \subset \oplus_{i \geq 2} \mathcal{O}_x(d_i) \oplus \mathcal{O}_x(C_1)$. On the other hand, $\text{Ker}(g_1)_x = \mathcal{O}_x(d_1)$, but since in_1 is always injective, we have that $\text{Ker}(\alpha_2)_x = \text{in}_1(\mathcal{O}(d_1))_x$.

Hence, if exists $x \in Z(f_1) \cap Z(g_1)$, then $\text{Ker}(\alpha_2)_x \cap \text{Im}(\alpha_1)_x = \emptyset$ and so the sequence (3) can not be exact in the middle, at x . Absurd.

Hence $Z(f_1) = \emptyset$ or $Z(g_1) = \emptyset$.

If $Z(f_1) = \emptyset$, then f_1 is never degenerate, so dualizing $E \xrightarrow{f_1} \mathcal{O}(d_1) \rightarrow 0$ we get $0 \rightarrow \mathcal{O}(d_1) \xrightarrow{(f_1)^T} E^*$, where the map $(f_1)^T$ is never degenerate; hence E^* splits, so E splits and X is a complete intersection.

If instead $Z(g_1) = \emptyset$, we build up the following commutative diagram:

$$\begin{array}{ccccccc}
& 0 & \longrightarrow & \mathcal{O}(d_1) & \xrightarrow{\equiv} & \mathcal{O}(d_1) & \longrightarrow \dots \\
& \downarrow & & \downarrow \text{in}_1 & & \downarrow g_1 & \\
0 & \longrightarrow & E & \longrightarrow & \bigoplus \mathcal{O}(d_i) \oplus \mathcal{O}(c_1) & \longrightarrow & C \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ker}(\psi) & \longrightarrow & \bigoplus_{i \geq 2} \mathcal{O}(d_i) \oplus \mathcal{O}(c_1) & \xrightarrow{\psi} & C' \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

By applying the snake lemma to the two central rows, we see that $\text{Ker}(\psi) \equiv E, C'$ is locally free since $Z(g_1) = \emptyset$, and we obtain a short exact sequence of *locally free* sheaves:

$$0 \longrightarrow E \longrightarrow \bigoplus_{i \geq 2} \mathcal{O}(d_i) \oplus \mathcal{O}(c_1) \longrightarrow C' \longrightarrow 0. \quad (4)$$

Repeating the previous reasoning, we can consider the morphisms $E \xrightarrow{f_2} \mathcal{O}(d_2)$ and $\mathcal{O}(d_2) \xrightarrow{g_2} C'$ and as before $Z(f_2) = \emptyset$ or $Z(g_2) = \emptyset$. If $Z(f_2) = \emptyset$, then E splits and X is a complete intersection. On the other hand, if $Z(g_2) = \emptyset$, arguing as before, we obtain a short exact sequence of locally free sheaves:

$$0 \longrightarrow E \longrightarrow \bigoplus_{i \geq 3} \mathcal{O}(d_i) \oplus \mathcal{O}(c_1) \longrightarrow C'' \longrightarrow 0.$$

In this way we obtain a sequence $Z(f_1), \dots, Z(f_{p-1})$. If one of these is empty, we are done; otherwise, if all are not empty, then, necessarily, $Z(g_{p-1}) = \emptyset$ and as before we obtain:

$$0 \longrightarrow E \longrightarrow \mathcal{O}(d_p) \oplus \mathcal{O}(c_1) \longrightarrow 0,$$

and we are done. \square

Proof of part B: To deal with the more restrictive case of characteristic zero, we can assume $k = \mathbb{C}$ in view of Lefschetz's principle. In our assumption

X is cut out schematically by n hypersurfaces V_1, \dots, V_n of degrees d_1, \dots, d_n and we have that $c_1(E) := a + b = d_k + b$ and $c_2(E) := ab = d_k b$ for some $k \in (1, \dots, n)$. (It is not restrictive to assume that $a \in (d_1, \dots, d_n)$). Reordering the hypersurfaces, we can assume that $c_1(E) = d_1 + b$ and $c_2(E) = d_1 b$. From the exact sequence (3), it is clear that the rank of C is $n - 1$, so that the morphism $\mathcal{O}(d_1) \xrightarrow{g_1} C$ degenerates at most in codimension $n - 1$. On the other hand, the morphism $E \xrightarrow{f_1} \mathcal{O}(d_1)$ can not degenerate in codimension 2, otherwise, if $Z(f_1)$ is its degeneracy locus, the Poincaré dual $[Z(f_1)] \in H^4(\mathbb{P}, \mathbb{Z})$ would represent $c_2(E^* \otimes \mathcal{O}(d_1)) = c_1(E^*)d_1 + c_2(E^*) + d_1^2 = 0$, so that either the morphism f_1 does not degenerate at all, and in this case we are done as before, or it degenerates in codimension 1.

So if the morphism f_1 degenerates in codimension one, we have that $Z(f_1) \cap Z(g_1) \neq \emptyset$, provided that $Z(g_1) \neq \emptyset$. On the other hand, by exactness of (3), we must have $Z(f_1) \cap Z(g_1) = \emptyset$, so that we conclude that $Z(g_1) = \emptyset$. Thus, arguing as in part A, we obtain the following short exact sequence:

$$0 \longrightarrow E \longrightarrow \bigoplus_{i \geq 2}^n \mathcal{O}(d_i) \oplus \mathcal{O}(c_1) \longrightarrow C' \longrightarrow 0,$$

and, from this, we conclude as in part A. \square

The result of part B of the previous theorem can be interpreted as a relation between degree $\deg(X)$ and subcanonicity e , recalling the well-known fact that if E is the vector bundle associated to X via Serre's correspondence, then $\deg(X) = c_2(E)$, while $e + n + 1 = c_1(E)$.

Corollary A: *Let $X \subset \mathbb{P}^n$ (X as in the hypotheses of Theorem B), $n \geq 3$ be scheme-theoretically defined by n hypersurfaces of degrees d_1, \dots, d_n , and let l be an integer in the set (d_1, \dots, d_n) . If the following relation is satisfied:*

$$\deg(X) + l^2 - (e + n + 1)l = 0, \quad (5)$$

then X is a complete intersection.

Proof: Arguing as in part B, it is clear that to show that E splits is sufficient to show that $c_2(E^* \otimes \mathcal{O}(l)) = 0$ for some l as above. But the vanishing of the second Chern class of $E^* \otimes \mathcal{O}(l)$ is exactly the relation (5), as an easy computation can show. \square

Remark 1: The approach of giving bounds on the degree of a subvariety to detect a complete intersection is particularly "effective", but it is obviously hopeless if one pretend to solve Hartshorne's conjecture. On the other hand,

since any closed subscheme (irreducible or not) of \mathbb{P}^n which is a local complete intersection is always scheme-theoretically defined by $n + 1$ hypersurfaces, as proved in [5], the approach of recognizing a complete intersection via the number of generators of its scheme-theoretic ideal, could be in principle useful to solve the conjecture. Unfortunately, the cases $p = n$ and in particular $p = n + 1$ (the generic case) appear completely intractable, at least up to now, since it is very difficult to relate the algebro-geometric properties of a small codimension embedding, with those of its scheme-theoretic ideal.

Remark 2: In the light of the previous remark and of Theorem B, it would be nice to know when a subvariety can be scheme-theoretically defined by n equations. To get a sufficient condition, we can use the theory of excess and residual intersections as developed in [3]. For example, let us consider 4 hypersurfaces $\{V_1, \dots, V_4\}$ in \mathbb{P}^4 , such that $\cap V_i = X \cup \{p_1, \dots, p_k\}$, where X is a smooth subcanonical surface and $\{p_1, \dots, p_k\}$ are (possibly non reduced) points, i.e. the four hypersurfaces define scheme-theoretically the union of X and a bunch of points outside X . The theory of residual intersections enables us to predict the (weighted) number of residual points as a function of the degrees of the hypersurfaces, of the degree of X and of the degrees of the Chern classes of T_X , the tangent sheaf of X . Imposing that the number of residual points is zero, gives us a sufficient condition for a surface (a subvariety in general) to be scheme-theoretically defined by n equations. Thus combining Proposition 9.12 (page 154) of [3] with Example 9.1.5. we get (for a surface in \mathbb{P}^4):

$$\begin{aligned} & \deg(c_2(T_X)) + (\sigma_1(g_i) - 5)\deg(c_1(T_X)) + \\ & + (\sigma_2(g_i) - 5\sigma_1(g_i) + 15)\deg(X) + W(p_1, \dots, p_k) = \sigma_4(g_i), \end{aligned} \quad (6)$$

where $W(p_1, \dots, p_k)$ is the weighted number of residual points and $\sigma_j(g_i)$ is the j -th elementary symmetric polynomial in the degrees g_i of the hypersurfaces V_i . Assuming X subcanonical, from $c_1(T_X) = -K_X$, we get $c_1(T_X) = -eH$, where H is the class of a hyperplane section; moreover, from the exact sequence:

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^4} \otimes \mathcal{O}_X \longrightarrow N_{X/\mathbb{P}^4} \longrightarrow 0,$$

we get $c_2(T_X) = 10H^2 + 5HK + K^2 - c_2(N_{X/\mathbb{P}^4})$, and by $c_1(T_X) = -eH$, we have $c_2(T_X) = (10 - 5e + e^2)H^2 - c_2(N_{X/\mathbb{P}^4})$. Since $\deg(c_2(N_{X/\mathbb{P}^4})) = d^2$,

$\deg(H^2) = d$ and $\deg(H) = d$, (where d is $\deg(X)$), substituting in (6), taking degrees and imposing $W(p_1, \dots, p_k) = 0$ reads:

$$[25 + \sigma_2(g_i) - (5 + e)\sigma_1(g_i) + e^2 - d]d = \sigma_4(g_i). \quad (7)$$

Thus, the relation (7) gives a sufficient condition for a subcanonical surface in \mathbb{P}^4 to be scheme-theoretically defined by 4 equations.

2.1 An application: the linkage criterion

As usual, if X and Y are l.c.i. of codimension 2 in \mathbb{P}^n , we say that X is (directly) linked to Y if there exists a complete intersection (F_1, F_2) , such that Y is the residual scheme of X in the intersection $F_1 \cap F_2$, and viceversa. In [1], working in characteristic zero and assuming X smooth and subcanonical $\dim X \geq 1$, Beorchia and Ellia proved that X is a complete intersection iff it is self-linked, i.e. iff there exists complete intersection (F_1, F_2) such that $F_1 \cap F_2 = 2X$ (F_1 and F_2 define on X a double structure which is a complete intersection). They also asked if the same criterion holds also for possibly singular l.c.i.. Recently, in [4], Franco, Kleiman and Lascu have given a positive answer to this question proving that the same criterion holds avoiding smoothness: X can be reducible and non reduced. Their proof works only in characteristic zero (unless $\dim X \geq 4$, where it holds over any algebraically closed field, due to a previous result of Faltings), so they ask if the same holds in positive characteristic, for lower dimensional X . Using our Theorem A we prove the following:

Proposition A: *Let X be a subcanonical (possibly singular) l.c.i. subscheme of codimension 2 in \mathbb{P}^n , $n \geq 4$, defined over an algebraically closed field of any characteristic. Then X is a complete intersection iff it is self-linked.*

Proof: According to the Gherardelli linkage theorem, which holds over any algebraically closed field (see [4] for its proof) we know that $X \subset F_1 \cap F_2$ is subcanonical iff its residual scheme Y (in the complete intersection $F_1 \cap F_2$) is scheme-theoretically defined by the intersection of F_1 and F_2 with a third hypersurface F_3 . On the other hand, if X is self-linked by F_1 and F_2 , then, by definition X is equal to its own residual scheme in the complete intersection of F_1 and F_2 , and since X is assumed subcanonical, by the Gherardelli theorem it is scheme theoretically defined by F_1, F_2 and F_3 ; hence, by Theorem A, it is a complete intersection as soon as $\dim X \geq 2$. Viceversa, if X is a complete

intersection, it is immediate to see that it is self-linked (just consider the intersection of F_1 and $2F_2$ if $X = F_1 \cap F_2$). \square

There is an immediate generalization of the previous proposition, which is the following:

Proposition B: *Let X as in Proposition A. Then X is a complete intersection iff it can be (directly) linked to Y , where Y is any subcanonical (possibly singular) l.c.i. subscheme.*

Proof: It is sufficient to use again the Gherardelli linkage and Theorem A. \square

Remark 3: The X 's as in Proposition B are self-linked iff they are scheme-theoretically defined by three hypersurfaces. Indeed, if X is self-linked, then by Gherardelli it is schematically defined by 3 equations; viceversa, if X is defined by 3 equations it is a complete intersection by Theorem A and then it is self-linked by Proposition B.

Remark 4: The most difficult case, in order to characterize complete intersection via self-linking is that of curves in \mathbb{P}^3 . Beorchia and Ellia proved their criterion (in characteristic zero) also for curves, assuming that they are smooth, while Franco, Kleiman and Lascu extended this result to l.c.i. curves (always working in characteristic zero). In positive characteristic (characteristic 2), however, there is certainly a counterexample for this criterion to hold in the case of curves, due to Migliore (see the discussion at the end of [4]). So our extension of this criterion over a field of any characteristic is the best possible for low dimensional subvarieties: that is surfaces are the lowest dimensional subvarieties where this criterion holds without exceptions. Unfortunately, up to now, there is no positive result for the case of space curves in characteristic greater than zero.

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